A CHARACTERIZATION OF SIMPLICITY OF REDUCED CROSSED PRODUCTS

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Abstract. Let G be a countable discrete group acting minimally on a compact Hausdorff space X. We show that if $C(X) \rtimes_r G$ is simple, then there is a comeager set of points of X with C*-simple stabilizer.

1. Introduction

Crossed products are a fundamental bridge between dynamics and operator algebras. Given a countable discrete group G acting on a compact space X, one can construct a C*-algebra $C(X) \rtimes_r G$ that encodes the dynamics. One of the most natural problems related to crossed products is to determine under which conditions a crossed product is simple. Minimality is a necessary condition, and is sufficient when combined with topological freeness. When the acting group is amenable, minimality and topological freeness of the action are equivalent to simplicity of $C(X) \rtimes_r G$ [1], but this characterization fails badly as soon as the amenable regime is left. Most notably, the trivial action of $\mathbb{F}_2$ on the one-point space gives rise to the simple algebra $\mathrm{C}^*_\mathrm{r}(\mathbb{F}_2)$. This interesting example found by Powers [2] gave rise to the study of C*-simple groups, that is, groups whose reduced C*-algebra is simple.

Simplicity of $C(X) \rtimes_r G$ is tightly connected to C*-simplicity of pointwise stabilizers of the action. Indeed, as shown by Ozawa in his lecture notes [3, Theorem 14], the existence of a point of X with C*-simple stabilizer is sufficient to ensure simplicity of $C(X) \rtimes G$. In the same notes, Ozawa asked whether the converse is true, and this is the main question investigated in these notes. Very recently, Hartman and Kalantar made significant progress on the question by showing that simplicity of $C(X) \rtimes_r G$ implies the existence of a point whose stabilizer has trivial amenable radical, thus completing the characterization for a broad class of groups [4].

The main result of this short note is a complete answer to Ozawa's question.

Theorem 1.1. *Let G be a countable discrete group acting minimally on a compact Hausdorff space X. Then the following are equivalent:*

(1) *The crossed product $C(X) \rtimes_r G$ is simple;*
(2) *there exists a point $x \in X$ such that the stabilizer G_x is* C**-simple;*
(3) *the set*
$$\{x \in X : G_x \textit{ is } \mathrm{C}^*\textit{-simple}\}$$
is comeager in X.

After completing a first draft of this work, we learned that Bray and Kennedy had already independently obtained the same result, and they were about to upload their solution to the arXiv. Their paper is now available [5], and they should be fully credited for finding the first solution. The two proofs are built around the same circle of ideas; thus these notes are not intended for publication. Most notably, Lemma 3.2, which is the key engine of the proof presented here, is essentially [5, Lemma 3.1]. The main—but still minor—difference in the two approaches is that the Baire category argument is made more explicit here. The authors of [5] have been informed that the solution was obtained independently, and they agreed on the uploading of this manuscript.

AI statement: ChatGPT has been used at an early stage of the project to review the literature on the subject and to discuss potential proof strategies. The author takes full responsibility for the content of the notes.

2. Preliminaries

2.1. **The Chabauty space.** Let G be a countable discrete group, and let $\mathrm{Sub}(G)$ be the subset of 2^G consisting of subgroups of G, identified with characteristic functions. The product topology turns 2^G into a compact metrisable space, and $\mathrm{Sub}(G)$ is a closed subset with respect to this topology.

For a finite subset $F \subseteq G \setminus \{e\}$, set

$$V_F := \{H \in \mathrm{Sub}(G) \colon H \cap F = \emptyset\}.$$

It is readily shown that the family of sets $(V_F)_{F \Subset G \setminus \{e\}}$ forms a local basis of neighborhoods of the trivial subgroup $\{e\}$.

We will use the following standard fact.

Lemma 2.1. *The set of amenable subgroups of G is closed in* $\mathrm{Sub}(G)$*. Moreover, the relation*

$$\{(H, K) \in \mathrm{Sub}(G)^2 \colon K \leq H\}$$

is closed.

For an action $G \curvearrowright X$, write

$$\mathrm{Stab} \colon X \longrightarrow \mathrm{Sub}(G), \qquad \mathrm{Stab}(x) = G_x := \{g \in G \colon gx = x\}.$$

This map need not be continuous, but it is Borel.

2.2. **Kennedy's and Kawabe's criteria.** We record two characterizations crucially used in the proof of Theorem 1.1. The first is Kennedy's characterization of C*-simplicity [7, Theorem 1.1], while the second is Kawabe's characterization of simplicity of crossed products [6, Theorem 6.1].

Theorem 2.2. *Let H be a discrete group. Then H is not C^*-simple if and only if there exist an amenable subgroup $K \leq H$ and a finite set $F \subseteq H \setminus \{e\}$ such that*

$$hKh^{-1} \cap F \neq \varnothing \qquad \textit{for every } h \in H.$$

Kawabe's theorem is the most natural crossed-product version of Kennedy's criterion.

Theorem 2.3. *Let G be a countable discrete group acting minimally on a compact space X. The following are equivalent:*

1. *$C(X) \rtimes_r G$ is simple;*
2. *for every $x \in X$ and every amenable subgroup $K \leq G_x$, there exists a sequence (g_n) in G such that*

$$g_n K g_n^{-1} \longrightarrow \{e\} \quad \textit{in } \mathrm{Sub}(G);$$

By the definition of the Chabauty topology, condition (2) is equivalent to

$$\forall x \in X \ \forall K \leq G_x \text{ amenable } \forall F \subseteq G \setminus \{e\} \text{ finite } \exists g \in G \colon \quad gKg^{-1} \cap F = \varnothing.$$

2.3. **The Furstenberg URS.** Let H be a countable group and let $\partial_F H$ denote its Furstenberg boundary. The Furstenberg boundary is defined as the set

$$\mathcal{A}_H = \{H_z \colon z \in \partial_F H\}.$$

Le Boudec and Matte Bon showed that every member of $\mathcal{A}_H$ is amenable, and the conjugation action $H \curvearrowright \mathcal{A}_H$ is a boundary action, see [8, Theorem 2.16 and Proposition 2.21].

We will also need functoriality under isomorphisms.

Lemma 2.4. *Let $\varphi\colon H \to H'$ be an isomorphism of countable groups. The induced homeomorphism*

$$\varphi_*\colon \mathrm{Sub}(H) \longrightarrow \mathrm{Sub}(H'), \qquad L \longmapsto \varphi(L),$$

satisfies

$$\varphi_*(\mathcal{A}_H) = \mathcal{A}_{H'}.$$

Proof. Transport the H-action on $\partial_F H$ through φ to an H'-action by setting

$$h' \cdot z = \varphi^{-1}(h') \cdot z.$$

This is a minimal strongly proximal H'-action. Its universal property is the same as that of the Furstenberg boundary of H', so it is H'-equivariantly homeomorphic to $\partial_F H'$. Under the transported action, the stabilizer of z is precisely $\varphi(H_z)$. □

3. The Furstenberg URS Lemma

For a finite subset $F \subseteq G \setminus \{e\}$, set

$$C_F = \left\{ H \in \mathrm{Sub}(G) \colon \begin{array}{l} \text{there exists an amenable } K \leq H \text{ such that} \\ hKh^{-1} \cap F \neq \emptyset \text{ for every } h \in H \end{array} \right\}. \tag{1}$$

Lemma 3.1. *For each finite subset $F \subseteq G \setminus \{e\}$ the set C_F is closed in* $\mathrm{Sub}(G)$.

Proof. Consider the set S_F of pairs $(H, K) \in \mathrm{Sub}(G)^2$ satisfying

(1) $K \leq H$,
(2) K is amenable,
(3) $hKh^{-1} \cap F \neq \emptyset$ for every $h \in H$.

Each of the above conditions is closed. Therefore S_F is closed, and hence compact. The set C_F is the projection of S_F on the first component, hence it is compact as well. □

The next lemma is the main property of the Furstenberg URS used in the proof of Theorem 1.1.

Lemma 3.2. *Let $H \leq G$ and let $F \subseteq G \setminus \{e\}$ be finite. If $H \in C_F$, then*

$$A \cap F \neq \varnothing \qquad \text{for every } A \in \mathcal{A}_H.$$

Proof. Since $H \in C_F$ there is an amenable $K \leq H$ such that for all $g \in H$ one has $gKg^{-1} \cap F \neq \emptyset$. Denote by Q_F the set of subgroups of H that intersect F. Then Q_F is closed in the Chabauty topology and $Q := \overline{\{gKg^{-1} \colon g \in H\}} \subseteq Q_F$. The set Q is closed, H-invariant, and all its elements are amenable subgroups of H. Let $K_0 \in Q$. Then $K_0 \curvearrowright \partial_F H$ and is amenable, therefore there is a K_0-invariant probability measure μ on $\partial_F H$. By strong proximality there is a net $(g_i)_i \subseteq H$ and an element $z \in \partial_F H$ such that $g_i\mu \to \delta_z$. Consider the net $g_i K_0 g_i^{-1}$ in Q. By compactness, up to passing to a subnet we can assume that $g_i K_0 g_i^{-1} \to K_1 \in Q$.

We claim that $K_1 \leq H_z$. Indeed, by definition of the Chabauty topology if $k \in K_1$, then $k \in g_i K_0 g_i^{-1}$ eventually, and therefore k fixes the measure $g_i\mu$. By continuity of the action on the space of probability measures, it follows that k fixes δ_z, and hence $kz = z$.

Let $A \in \mathcal{A}_H$. Since the action $H \curvearrowright \mathcal{A}_H$ is minimal, there is a net $(h_i) \subseteq H$ such that $h_i H_z h_i^{-1} \to A$. Since $(h_i K_1 h_i^{-1})$ is a net in Q, up to passing to a subnet we can assume that $h_i K_1 h_i^{-1} \to K_2$ for some $K_2 \in Q$. Since being a subgroup is a closed condition, $K_2 \leq A$. Moreover $K_2 \cap F \neq \emptyset$ since $K_2 \in Q_F$, hence $A \cap F \neq \emptyset$. □

4. Proof of the main theorem

Let

$$B = \{x \in X \colon G_x \text{ is not } C^*\text{-simple}\},$$

and for finite $F \subseteq G \setminus \{e\}$ set

$$B_F = \mathrm{Stab}^{-1}(C_F) = \{x \in X \colon G_x \in C_F\}. \tag{2}$$

Kennedy's characterization of C*-simplicity readily gives the inclusion $B \subseteq \bigcup_F B_F$. Therefore the implication (3)⇒(1) in Theorem 1.1 reduces to the proof that each set B_F is meager, under that assumption that $C(X) \rtimes_r G$ is simple. To show meagerness, we prove some basic properties of the sets B_F.

A subset S of a topological space X has the Baire property if it can be written as the symmetric difference of an open and a meager set. The family of sets with the Baire property is a σ-algebra, and it trivially contains all open sets. Therefore every Borel set has the Baire property.

Lemma 4.1. *For every finite $F \subseteq G \setminus \{e\}$, the set B_F is Borel in X, hence has the Baire property. Moreover:*

(1) *if $F \subseteq F'$, then $B_F \subseteq B_{F'}$;*
(2) *for every $g \in G$,*

$$gB_F = B_{gFg^{-1}}.$$

Proof. Since Stab is a Borel map and C_F is closed, B_F is Borel. Monotonicity in F immediately follows from the definition of C_F.

Suppose $x \in B_F$ and choose an amenable $K \leq G_x$ witnessing $G_x \in C_F$. Then gKg^{-1} is an amenable subgroup of

$$G_{gx} = gG_x g^{-1},$$

and for $h' = ghg^{-1} \in G_{gx}$ one has

$$h'(gKg^{-1})(h')^{-1} \cap gFg^{-1} = g(hKh^{-1} \cap F)g^{-1} \neq \emptyset.$$

Thus $gx \in B_{gFg^{-1}}$, so $gB_F \subseteq B_{gFg^{-1}}$. Applying the same argument to g^{-1} gives equality. □

The following Lemma allows us to improve non-meagerness of a set B_F to co-meagerness of B_{F_0} for some other F_0.

Lemma 4.2. *If B_F is nonmeager for some finite F, then there exists a finite set $F_0 \subseteq G \setminus \{e\}$ such that B_{F_0} is comeager in X.*

Proof. The set B_F has the Baire property by Lemma 4.1. Since B_F is non-meager, there is a nonempty open set $U \subseteq X$ such that B_F is comeager in U. By compactness of X and minimality of the action, there are elements $g_1, \dots g_n \in G$ such that $X = g_1 U \cup \dots \cup g_n U$. Therefore for each j, the set $B_{g_j F g_j^{-1}} = g_j B_F$ is comeager in $g_j U$, and therefore B_{F_0} is comeager in X for $F_0 = \bigcup_{j=1}^n g_j F g_j^{-1}$. □

We are now ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. The implication (2)⇒(1) is Ozawa's result [3, Theorem 14], while (3)⇒(2) is trivial. We show that (1)⇒(3) by showing that each set B_F is meager.

Suppose this is not the case. Then by Lemma 4.2 there is a finite $F_0 \subseteq G \setminus \{e\}$ for which B_{F_0} is comeager. Let

$$Y = \bigcap_{g \in G} gB_{F_0}.$$

The set Y is comeager, and it is G-invariant. Let $x \in Y$ and $A \in \mathcal{A}_{G_x}$. By Lemma 2.4, the amenable subgroup gAg^{-1} is an element of $\mathcal{A}_{gG_xg^{-1}} = \mathcal{A}_{G_{gx}}$. Since $gx \in Y$, the pointwise stabilizer $G_{gx} \in B_{F_0}$. Applying Lemma 3.2 to G_{gx}, F_0, and gAg^{-1}, one obtains

$$gAg^{-1} \cap F_0 \neq \emptyset$$

for all $g \in G$. On the other hand, Theorem 2.3 and simplicity of $C(X) \rtimes_r G$ forces the existence of $g \in G$ so that $gAg^{-1} \cap F_0 = \emptyset$, giving the desired contradiction. □

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